SOLUTION BY FRANK IRWIN, University of California.

On the right side

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\binom{n+1}{n-k+1}.$$

This formula expresses in symbols the well-known property of the Pascal triangle, that any term is equal to the sum of all terms above it in the preceding column. (See, for instance, LUCAS, *Théorie des Nombres*, page 6.)

Or it may be proved as follows:

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\sum_{i=0}^{k}\left[\binom{n-i+1}{n-k+1}-\binom{n-i}{n-k+1}\right]=\sum_{i=0}^{k}\binom{n-i+1}{n-k+1}-\sum_{i=1}^{k+1}\binom{n-i+1}{n-k+1}.$$

Here all the terms but one cancel in pairs, leaving $\binom{n+1}{n-k+1}$.

On the other hand, the left side of our given equation may be written

$$\frac{2^k}{\lfloor\!\underline{n-k+1}\,\lfloor\!\underline{k}}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\binom{n-k+1}{2i+1}.$$

Here the summation sign takes in the 2d, 4th, $\cdots$, *all* the even-placed coefficients,

$$\binom{n-k+1}{1},\quad\binom{n-k+1}{3},\ \cdots$$

in the development of $(1+x)^{n-k+1}$; the sum of which is well known to be 2^{n-k}.

Our formula, then, reduces to the obviously true form:

$$\frac{1}{\lfloor\!\underline{n-k+1}}\,\frac{2^k}{\lfloor\!\underline{k}}\,2^{n-k}=\frac{2^n}{\lfloor\!\underline{n+1}}\binom{n+1}{n-k+1},$$

Also solved by the Proposer.

440. Proposed by W. D. CAIRNS, Oberlin College.

n being a positive integer, find the sum of the series

$$2n^2+4(n-1)^2+2(n-2)^2+4(n-3)^2+2(n-4)^2+\cdots,$$

where the succeeding coefficients are alternately 4 and 2; or, more generally, the series

$$an^2+b(n-1)^2+a(n-2)^2+b(n-3)^2+a(n-4)^2+\cdots.$$

L'Intermédiaire, July, 1913.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The problem as originally published in the September, 1915, issue contained two misprints. The question is indefinite. These series are not convergent, and do not break off after a finite number of terms. However, it is easy to find the sum of $2k$ terms.

Expanding, we can write the second series as

$$\begin{aligned}S&=a\{n^2+n^2-4n+4+n^2-8n+16+n^2-12n+36+\cdots+n^2-2(2k-2)n+(2k-2)^2\}\\&\quad+b\{n^2-2n+1+n^2-6n+9+n^2-10n+25+\cdots+n^2-2(2k-1)n+(2k-1)^2\}\\&=a\{kn^2-4n(1+2+3+\cdots+k-1)+(2^2+4^2+6^2+\cdots+[2k-2]^2)\}\\&\quad+b\{kn^2-2n(1+3+5+\cdots+2k-1)+(1^2+3^2+5^2+\cdots+[2k-1]^2)\}.\end{aligned}$$

Summing the series in parentheses we obtain

$$S = a\left\{ kn^2 - 2nk(k-1) + \frac{4k^3 - 6k^2 + 2k}{3} \right\} + b\left\{ kn^2 - 2nk^2 + \frac{4k^3 - k}{3} \right\},$$

which may be written as

$$(a+b)kn^2 - 2nk(ak + bk - a) + \left\{ \frac{4}{3}k^3(a+b) - 2ak^2 + \frac{k}{3}(2a - b) \right\}.$$

Also solved by H. C. FEEMSTER, HORACE OLSON and the PROPOSER.

441. Proposed by W. D. CAIRNS, Oberlin College.

Prove that the equation $(e-1)x = e^x - 1$ has two and only two real roots.

I. SOLUTION BY H. S. UHLER, Yale University.

Let $y = e^x - 1 - (e-1)x$ and observe that $y = 0$ for $x = 0$ and $x = 1$. It remains to show that there can be no more real roots.

$$\frac{dy}{dx} = e^x - e + 1, \quad (1) \qquad \frac{d^2y}{dx^2} = e^x. \quad (2)$$

Equation (1) shows that the slope of the tangent is positive for all values of x greater than $\log_e(e-1)$ and negative for all values of x less than this value ($x_0 = 0.541325$). Since e^x is essentially positive, equation (2) indicates formally that the single stationary point, indicated by equation (1), corresponds to a minimum value of y. The coördinates of the minimum are $x_0 = \log_e(e-1)$ and $y_0 = (e-1)(1-x_0) - 1 = -0.211867$. It is clear, therefore, from the properties of the graph that the curve cannot cut the axis of x in more than two points and hence the given equation has two and only two real roots.

II. SOLUTION BY GRACE M. BAREIS, Ohio State University.

Writing e in series form in each member and transposing, the equation becomes

$$x\left[\frac{x-1}{\underline{|2}} + \frac{x^2-1}{\underline{|3}} + \frac{x^3-1}{\underline{|4}} + \cdots\right] = 0, \text{ or } x(x-1)\left[\frac{1}{\underline{|2}} + \frac{x+1}{\underline{|3}} + \frac{x^2+x+1}{\underline{|4}} + \cdots\right] = 0.$$

Hence, $x = 0$, $x - 1 = 0$, or $\frac{1}{\underline{|2}} + \frac{x+1}{\underline{|3}} + \frac{x^2+x+1}{\underline{|4}} + \cdots = 0$. Since each term of the left member of the last equation is positive, this equation can have no real positive root.

To prove that the original equation can have no real negative root, put it in the form

$$x = \frac{e^x - 1}{e - 1},$$

or $x = e^{x-1} + e^{x-2} + e^{x-3} + \cdots$, a convergent series. Any real negative number makes the left member negative but the right member positive, and hence there are no real negative roots. Hence, 0 and 1 are the only real roots.

Solutions were also received from W. L. AGARD, F. L. GRIFFIN, H. C. FEEMSTER, WALTER C. EELLS, HORACE OLSON, C. E. HORNE, IRBY C. NICHOLS, and FRANK IRWIN.

GEOMETRY.

467. Proposed by E. T. BELL, Seattle, Washington.

It is well-known that if i, j, k, l are concyclic points, W_i the Wallace line (frequently, and erroneously, called the Simson line), of i with respect to the triangle jkl, then W_i, W_j, W_k, W_l are concurrent, say in the point $\{i, j, k, l\}$. If 1, 2, 3, $\cdots$ denote concyclic points, prove that:

(i) $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$ are concyclic; say on the circle [1, 2, 3, 4, 5];

(ii) Starting with 1, 2, 3, 4, 5, 6, omitting each point in turn, by (i), six circles, are found; these are concurrent, say in the point $\{1, 2, 3, 4, 5, 6\}$;

(iii) Starting with 1, 2, 3, 4, 5, 6, 7, seven points of the kind in (ii) are found; these lie on a circle.